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Generalization of the Darboux transformation and generalized harmonic oscillators

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Abstract

The Darbroux transformation is generalized for time-dependent Hamiltonian systems which include a term linear in momentum and a time-dependent mass. The formalism for the *N*-fold application of the transformation is also established, and these formalisms are applied for a general quadratic system (a generalized harmonic oscillator) and a quadratic system with an inverse-square interaction up to $N = 2$. Among the new features found, it is shown, for the general quadratic system, that the shape of potential difference between the original system and the transformed system could oscillate according to a classical solution, which is related to the existence of coherent states in the system.

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1. Introduction

There has been great interest in using the Darboux transformation [1, 2] for the analysis of physical systems and for finding new solvable systems. It has been shown that the transformation method is useful in finding soliton solutions of the integrable systems [3] and constructing supersymmetric quantum mechanical systems [4]. An excellent survey of developments and some applications of the transformation method are given in [5].

If a Darboux transformation is applied to a time-independent Schrödinger equation of confining potential, one of the immediate consequences is that the energy eigenvalues of the transformed system will be almost identical to those of the original system except for a finite number of addition(s) and/or deletion(s) to the spectrum [6]. On the other hand, Abraham and Moses (AM) have developed an integral equation algorithm [7], which can also be used to find a new solvable system based on a known one. When applied to a harmonic oscillator, the AM algorithm gives a transformed system whose energy spectrum coincides with that of the harmonic oscillator except that the lowest eigenvalue has been removed. It has been shown that the result given by AM can also be derived through the factorization method [8], and the factorization method has been applied for various systems [9]. After all, the AM method and the factorization method are intimately related to the Darboux transformation [6, 10, 11].

In order to obtain a new solvable system by implementing the Darboux transformation, it is necessary to choose the auxiliary function of the transformation judiciously [6]. After the transformation is extended to include a time-dependent potential, the application of the Darboux transformation and the two-fold application of the transformation have been explicitly carried out for the harmonic oscillator model (with time-dependent frequency), to obtain new solvable systems [11, 12]. For the simple harmonic oscillator, unphysical negative-energy eigenstates, which can be obtained by invoking the symmetry of the Hamiltonian, have been used as the auxiliary functions of the Darboux transformation. For the two-fold application of the transformation, a theorem has already been established in [13] for the choice of the auxiliary functions.

In this paper, we will generalize the Darbroux transformation to be applicable to time-dependent Hamiltonian systems which include a term linear in momentum and a time-dependent mass. The formalism for the *N*-fold application of the transformation will also be established. The transformation method will then be applied for a general quadratic system (a generalized harmonic oscillator) and a quadratic system with an inverse-square interaction up to $N = 2$. The general quadratic system [14] is known to be related to a simple harmonic oscillator through unitary transformations, which may be a reason for the existence of coherent states in the simple harmonic oscillator [15]. Even for the cases already considered [11, 12], new features of the Darboux transformed systems will be given. In particular, it will be shown that the shape of the potential difference between the original Hamiltonian and the transformed one could oscillate according to the classical solution of the quadratic system, which is related to the existence of coherent states in the system.

2. A generalization of the Darboux transformation

For smooth functions $P(x)$, $Q(x)$, we assume a function $u(x)$ satisfies

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}u}{\mathrm{d}x} + (Q(x) + C)u = 0 \tag{1}$$

where C is a constant. If $\phi(x)$ satisfies the linear differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}\phi}{\mathrm{d}x} + Q(x)\phi = 0 \tag{2}$$

Darboux shows that the following equation is true [1]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}P}{\mathrm{d}x} + Q(x) + 2\left(\frac{\mathrm{d}^2\ln u}{\mathrm{d}x^2}\right)\right]\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} - \phi\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0. \tag{3}$$

This Darboux transformation has immediate consequences for a time-independent Schrödinger equation [6].

For the extension of the transformation to be applicable for the time-dependent system described by the Hamiltonian

$$H = \frac{p^2}{2M(t)} + (R(x,t)p + pR(x,t)) + V(x,t) \tag{4}$$

we consider the operator

$$\begin{aligned} O(t,x) &= -\mathrm{i}\hbar\frac{\partial}{\partial t} + H \\ &= -\mathrm{i}\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2M(t)}\frac{\partial^2}{\partial x^2} - \mathrm{i}\hbar\left\{2R(x,t)\frac{\partial}{\partial x} + R'(x,t)\right\} + V(x,t). \end{aligned} \tag{5}$$

In H, $M(t)$ denotes time-dependent mass and the term proportional to $R(x,t)$ is included to resemble a three-dimensional model which is under a vector potential. A wavefunction of the system of H should satisfy the time-dependent Schrödinger equation

$$O(t,x)\psi(t,x) = 0. \tag{6}$$

We also introduce the auxiliary functions u_k satisfying

$$O(t,x)u_k(t,x) = 0 \qquad (k = 1, 2, \ldots, N). \tag{7}$$

We assume that u_k is a smooth function for any finite x, but u_k need not be square-integrable as it is just an auxiliary function.

From (6) and (7), it is straightforward to show that

$$\left(O(t,x) - 2\mathrm{i}\hbar R'(x,t) - \frac{\hbar^2}{M(t)}\frac{\partial^2 \ln u_1}{\partial x^2}\right)\psi^{1d}(x,t) = 0 \tag{8}$$

where

$$\psi^{1d}(x,t) = \left(\psi' - \psi\frac{u_1'}{u_1}\right) \tag{9}$$

and $'$ denotes the partial derivative with respect to x. Equation (8) is a generalization of the Darboux transformation.

The Darboux transformation can be applied repeatedly. For the description of k-fold transformations, we define the Wronskian determinants as

$$W_k = \begin{vmatrix} u_1 & u_2 & \cdots & u_k \\ u_1' & u_2' & \cdots & u_k' \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(k-1)} & u_2^{(k-1)} & \cdots & u_k^{(k-1)} \end{vmatrix} \qquad W_{k,\psi} = \begin{vmatrix} u_1 & u_2 & \cdots & u_k & \psi \\ u_1' & u_2' & \cdots & u_k' & \psi' \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_k^{(k)} & \psi^{(k)} \end{vmatrix}. \tag{10}$$

Making use of Crum's formula [2]

$$W_{k,\psi}W_{k-1} = W_k\frac{\partial}{\partial x}W_{k-1,\psi} - W_{k-1,\psi}\frac{\partial}{\partial x}W_k \tag{11}$$

one can find that $\psi^{kd}(x,t)$ satisfying

$$\left(O(t,x) - 2\mathrm{i}k\hbar R'(x,t) - \frac{\hbar^2}{M(t)}\frac{\partial^2 \ln W_k}{\partial x^2}\right)\psi^{kd}(x,t) = 0 \tag{12}$$

is given by

$$\psi^{kd}(x,t) = \frac{W_{k,\psi}}{W_k}. \tag{13}$$

There is a difficulty in interpreting equation (12) as the Schrödinger equation, since the associated Hamiltonian is, in general, not Hermitian. As discussed in special cases [11], this difficulty can be resolved if there exists a purely time-dependent function $\alpha_k(t)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\alpha_k = 2kR' - \mathrm{i}\frac{\hbar}{2M(t)}\frac{\partial^2}{\partial x^2}\ln\frac{W_k}{\bar{W}_k} \tag{14}$$

where $\bar{W}_k$ denotes the complex conjugate of W_k. In this case, equation (12) can be rewritten as

$$\left(O(t,x) - \frac{\hbar^2}{2M(t)} \frac{\partial^2 (W_k \bar{W}_k)}{\partial x^2} \right) \psi^{kD}(x,t) = 0 \tag{15}$$

where

$$\psi^{kD}(x,t) = \alpha_k(t) \psi^{kd}(x,t). \tag{16}$$

If $V(x,t)$ is a real function, equation (15) is the Schrödinger equation of a system described by the Hermitian Hamiltonian

$$H_k = \frac{p^2}{2M(t)} + (R(x,t)p + pR(x,t)) + V(x,t) - \frac{\hbar^2}{2M(t)} \frac{\partial^2 \ln(W_k \bar{W}_k)}{\partial x^2}. \tag{17}$$

One of the crucial conditions for $\psi^{kD}(x,t)$ being square-integrable is that W_k should not have any zero in the entire space of x. Indeed, the derivations of (12) and (13) are not valid for the zeros of $W_1, W_2, \ldots, W_k$, while, if W_k has no zero, the formulae would still be useful even for the cases that $W_1, W_2, \ldots, W_{k-1}$ have zeros, as examples will show later.

In addition to $\psi^{kD}(x,t)$, other solutions of the Schrödinger equation of H_k would exist [6]. For the H_1 system, from the fact that

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \frac{1}{u} = \frac{\hbar^2}{2M} \left(\frac{\partial^2}{\partial x^2} \frac{1}{u} \right) - 2\mathrm{i}\hbar R \left(\frac{\partial}{\partial x} \frac{1}{u} \right) + (-V + \mathrm{i}\hbar R') \frac{1}{u} + \frac{1}{u} \frac{\hbar^2}{M} \left(\frac{\partial^2 \ln u}{\partial x^2} \right) \tag{18}$$

a solution ψ_u^{1D} satisfying $[-\mathrm{i}\hbar(\partial/\partial t) + H_1]\psi_u^{1D} = 0$ is given as [6, 11]

$$\psi_u^{1D}(x,t) = \frac{1}{\alpha \bar{u}_1} = \frac{1}{\alpha \bar{W}_1}. \tag{19}$$

3. A general quadratic system

For an application of the transformation of the previous section, it may be essential to find u_1 or W_k which does not vanish over the whole coordinate space. Moreover, for an Hermitian Hamiltonian, such auxiliary solutions are required to support the existence of a purely time-dependent function, $\alpha_k(t)$, defined in (14). For the non-vanishing u_1 of the harmonic oscillator, unphysical negative-energy eigenstates can be used as the auxiliary functions for the transformation. For the two-fold application, the sign theorem established in [13] may be used to find non-vanishing W_2. Nevertheless, it has been shown that the application and two-fold application of the Darboux transformation are possible for the harmonic oscillator (of time-dependent frequency) with and without an inverse-square potential [11, 12].

In this section, we will show that the application and two-fold application of the Darboux transformation are possible for a general quadratic system. Even for the cases already considered [11, 12], new features of the applications will be found.

3.1. Darboux transformation

A general quadratic system can be conveniently described by the Lagrangian [14]

$$L_Q = \frac{1}{2} M(t) \dot{x}^2 - \frac{1}{2} M(t) w^2(t) x^2 + F(t) x + \frac{\mathrm{d}}{\mathrm{d}t}(M(t) a(t) x^2) + \frac{\mathrm{d}}{\mathrm{d}t}(b(t) x) + f(t) \tag{20}$$

where the overdot denotes the derivative with respect to t. This Lagrangian gives the classical equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(t)\dot{x}) + M(t) w^2(t) x = F(t) \tag{21}$$

which is that of a generalized harmonic oscillator of mass $M(t)$ and frequency $w(t)$ in an external force $F(t)$. The corresponding Hamiltonian is written as

$$H_Q = \frac{p^2}{2M(t)} - a(t)[px + xp] + \frac{1}{2}M(t)c(t)x^2 - \frac{b(t)}{M(t)}p + d(t)x + \left(\frac{b^2(t)}{2M(t)} - f(t)\right) \tag{22}$$

where

$$c(t) = w^2 + 4a^2 - 2\dot{a} - 2\frac{\dot{M}}{M}a \qquad d(t) = 2ab - \dot{b} - F. \tag{23}$$

We assume that H_Q is Hermitian. The general solution of equation (21) is a linear combination of a particular solution $x_p(t)$ and two linearly independent homogeneous solutions $u(t)$, $v(t)$. We assume $x_p(t)$, $u(t)$, $v(t)$ are real, and define $\rho(t)$ and a time constant Ω, for later use, as

$$\rho = \sqrt{u^2(t) + v^2(t)} \qquad \Omega = M(t)[\dot{v}(t)u(t) - \dot{u}(t)v(t)]. \tag{24}$$

Since the general quadratic system can be obtained from a simple harmonic oscillator through unitary transformations, we first consider the simple harmonic oscillator of unit mass and frequency. The Schrödinger equation for the simple harmonic oscillator, whose time is τ, is written as

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\psi_s = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\psi_s + \frac{x^2}{2}\psi_s = H_s\psi_s. \tag{25}$$

Making use of the invariance of the Schrödinger equation under the exchange of $\tau \leftrightarrow -\tau$ and $x \leftrightarrow \mathrm{i}x$, from the well-known wavefunctions

$$\psi_n^s(\tau, x) = \frac{\mathrm{e}^{-\mathrm{i}(n+1/2)\tau}}{(2^n n!\sqrt{\pi\hbar})^{1/2}}\exp\left[-\frac{x^2}{2\hbar}\right]H_n\left(\frac{x}{\sqrt{\hbar}}\right) \tag{26}$$

one can find auxiliary functions satisfying the same Schrödinger equation as

$$v_n^s(\tau, x) = \mathrm{e}^{\mathrm{i}(n+1/2)\tau}\exp\left[\frac{x^2}{2\hbar}\right]H_n\left(\frac{\mathrm{i}x}{\sqrt{\hbar}}\right) \qquad n = 0, 1, 2, \ldots. \tag{27}$$

Indeed, $v_n^s(\tau, x)$ for even n has no zero over the whole coordinate space, and it can be used as an auxiliary function of the Darboux transformation. In this subsection, we restrict our attention to the cases where $n = 0, 2, 4, \ldots$.

By defining

$$O_s(\tau, x) = -\mathrm{i}\hbar\frac{\partial}{\partial\tau} + H_s \qquad O_Q(t, x) = -\mathrm{i}\hbar\frac{\partial}{\partial t} + H_Q \tag{28}$$

and if τ and t are related by

$$\mathrm{d}\tau = \frac{\Omega}{M(t)\rho^2(t)}\mathrm{d}t \tag{29}$$

then from the results given in [15] one can find that

$$O_Q(t, x) = U_G U_F U_S O_s(\tau, x) U_S^\dagger U_F^\dagger U_G^\dagger|_{\tau=\tau(t)} \tag{30}$$

where

$$U_G = \exp\left[\frac{\mathrm{i}}{\hbar}\left(M(t)a(t)x^2 + b(t)x + \int^t f(z)\,\mathrm{d}z\right)\right] \tag{31}$$

$$U_F(x_p) = \exp\left[\frac{\mathrm{i}}{\hbar}(M\dot{x}_p x + \delta(x_p))\right]\exp\left(-\frac{\mathrm{i}}{\hbar}x_p p\right) \tag{32}$$

$$U_S(\rho, \Omega) = \exp\left[\frac{\mathrm{i}}{2\hbar}M\frac{\dot{\rho}}{\rho}x^2\right]\exp\left[-\frac{\mathrm{i}}{4\hbar}\ln\left(\frac{\rho^2}{\Omega}\right)(xp + px)\right] \tag{33}$$

with δ defined through the relation

$$\dot{\delta}(x_p) = \tfrac{1}{2}Mw^2x_p^2 - \tfrac{1}{2}M\dot{x}_p^2. \tag{34}$$

A point that should be mentioned is the unitary operators U_F and U_S are not unique since they depend on the choice of classical solutions. Instead of $\{u, v\}$ and x_p, one can take another set of two linearly independent homogeneous solutions $\{\tilde{u}, \tilde{v}\}$ and a particular solution $\tilde{x}_p$ of equation (21). After defining $\tilde{\rho}$, $\tilde{\Omega}$ and $\tilde{\delta}$ from $\{\tilde{u}, \tilde{v}\}$ and $\tilde{x}_p$ as ρ, Ω and δ are defined from $\{u, v\}$ and x_p, one can find that the unitary relation (30) is also valid with the unitary operators $U_F(\tilde{x}_p)$, $U_S(\tilde{\rho}, \tilde{\Omega})$.

Making use of the unitary relation, the normalized wavefunctions of the system of H_Q have been given in [15] as

$$\begin{aligned}\psi_m^Q(t,x) = {} & \frac{1}{\sqrt{2^m m!}}\left(\frac{\Omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\rho(t)}}\left[\frac{u(t) - \mathrm{i}v(t)}{\rho(t)}\right]^{m+\frac{1}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\left(\delta(t) + \int^t f(z)\,\mathrm{d}z\right)\right] \\ & \times \exp\left[\frac{\mathrm{i}}{\hbar}[M(t)a(t)x^2 + (M(t)\dot{x}_p(t) + b(t))x]\right] \\ & \times \exp\left[\frac{(x - x_p(t))^2}{2\hbar}\left(-\frac{\Omega}{\rho^2(t)} + \mathrm{i}M(t)\frac{\dot{\rho}(t)}{\rho(t)}\right)\right] H_m\left(\sqrt{\frac{\Omega}{\hbar}}\frac{x - x_p(t)}{\rho(t)}\right).\end{aligned} \tag{35}$$

The unitary relation can also be used to find a solution $v_n^Q(t, x)$ of the Schrödinger equation

$$O(t,x)v_n^Q(t,x) = 0 \tag{36}$$

as

$$\begin{aligned}v_n^Q(t,x) &= U_G U_F(\tilde{x}_p) U_S(\tilde{\rho}, \tilde{\Omega}) v_n^s(\tau, x)|_{\tau=\tau(t)} \\ &= \sqrt{\frac{\sqrt{\tilde{\Omega}}}{\tilde{\rho}(t)}}\left[\frac{\tilde{u}(t) + \mathrm{i}\tilde{v}(t)}{\tilde{\rho}(t)}\right]^{n+\frac{1}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\left(\tilde{\delta}(t) + \int^t f(z)\mathrm{d}z\right)\right] \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}[M(t)a(t)x^2 + (M(t)\dot{\tilde{x}}_p(t) + b(t))x]\right] \\ &\quad \times \exp\left[\frac{(x - \tilde{x}_p(t))^2}{2\hbar}\left(\frac{\tilde{\Omega}}{\tilde{\rho}^2(t)} + \mathrm{i}M(t)\frac{\dot{\tilde{\rho}}(t)}{\tilde{\rho}(t)}\right)\right] H_n\left(\mathrm{i}\sqrt{\frac{\tilde{\Omega}}{\hbar}}\frac{x - \tilde{x}_p(t)}{\tilde{\rho}(t)}\right).\end{aligned} \tag{37}$$

From the properties of the unitary transformations, it is manifest that $v_n^Q(t, x)$ for even n does not have a zero over whole coordinate space.

Since, for even n, $H_n\big(\mathrm{i}\sqrt{\frac{\tilde{\Omega}}{\hbar}}\frac{x-\tilde{x}_p(t)}{\tilde{\rho}(t)}\big)$ is a real function, it is easy to find that

$$-\mathrm{i}\frac{\hbar}{2M}\frac{\partial^2}{\partial x^2}\ln\frac{v_n^Q}{\bar{v}_n^Q} = 2a + \frac{\dot{\tilde{\rho}}}{\tilde{\rho}} \tag{38}$$

which shows, for the choice of $u_1^Q = W_1^Q = v_n^Q$, that $\alpha_1^Q(t)$ defined in (14) can be found as

$$\alpha_1^Q(t) = \tilde{\rho}(t) \tag{39}$$

up to a normalization constant. Therefore, the transformation formalism developed in the previous section can be applied with $u_1^Q(t, x) = v_n^Q(t, x)$ to give the solvable model described by the Hamiltonian

$$H_1^{nQ}(t,x,p) = H_Q - \frac{\hbar\tilde{\Omega}}{M\tilde{\rho}^2} + 4n\frac{\hbar\tilde{\Omega}}{M\tilde{\rho}^2}\left[(n-1)\frac{H_{n-2}(z)}{H_n(z)} - n\left(\frac{H_{n-1}(z)}{H_n(z)}\right)^2\right] \tag{40}$$

where $z = \mathrm{i}\sqrt{\frac{\tilde{\Omega}}{\hbar}}\frac{x-\tilde{x}_p}{\tilde{\rho}}$. If we adopt the notation that $H_{-2}(z) = H_{-1}(z) = 0$, equation (40) is valid for $n = 0, 2, 4, \ldots$. The magnitude of ΔV $\left(\equiv H_1^{nQ} - H_Q + \frac{\hbar\tilde{\Omega}}{M\tilde{\rho}^2}\right)$ is $O(\hbar)$. Since ΔV vanishes in the limit of $|z| \to \infty$, $\Delta V/\hbar$ approaches 0 except for the region in the vicinity of x_p where the width of the region is $O(\sqrt{\hbar})$.

From equations (10), (13) and (16), the unnormalized wavefunctions $\psi_m^{nQ}(t,x)$ satisfying the Schrödinger equation

$$\left(-\mathrm{i}\hbar\frac{\partial}{\partial t} + H_1^{nQ}\right)\psi_m^{nQ}(t,x) = 0 \tag{41}$$

can be given as

$$\psi_m^{nQ}(t,x) = \tilde{\rho}\psi_m\left[\frac{\mathrm{i}}{\hbar}M(\dot{x}_p - \dot{\tilde{x}}_p) + \frac{x-x_p}{\hbar}\left(-\frac{\Omega}{\rho^2} + \mathrm{i}M\frac{\dot{\rho}}{\rho}\right) - \frac{x-\tilde{x}_p}{\hbar}\left(\frac{\tilde{\Omega}}{\tilde{\rho}^2} + \mathrm{i}M\frac{\dot{\tilde{\rho}}}{\tilde{\rho}}\right) + 2m\sqrt{\frac{\Omega}{\hbar}}\frac{1}{\rho}\frac{H_{m-1}(w)}{H_m(w)} - 2\mathrm{i}n\sqrt{\frac{\tilde{\Omega}}{\hbar}}\frac{1}{\tilde{\rho}}\frac{H_{n-1}(z)}{H_n(z)}\right] \tag{42}$$

where $w = \sqrt{\frac{\Omega}{\hbar}}\frac{x-x_p}{\rho}$. As in the general quadratic system [14, 15], a different choice of the classical solutions $u(t), v(t), x_p(t)$ gives different wavefunctions. When we choose $u(t) = \tilde{u}(t)$, $v(t) = \tilde{v}(t)$ and $x_p(t) = \tilde{x}_p(t)$, for even integer n, an unnormalized $\psi_m^{nQ}(t,x)$ is written as

$$\psi_m^{nQ}(t,x) = -\sqrt{\frac{2\Omega}{\hbar}}\left[\sqrt{m+1}\left(\frac{u+\mathrm{i}v}{\rho}\right)\psi_{m+1} + \sqrt{2}n\mathrm{i}\frac{H_{n-1}(\mathrm{i}w)}{H_n(\mathrm{i}w)}\psi_m\right] \tag{43}$$

for $m = 0, 1, 2, \ldots$. For the case of $M(t) = 1$ and $a(t) = 0$, $\psi_m^{2Q}(t,x)$ in equation (43) reproduces the wavefunctions given in [12], up to normalization. From equation (19), another wavefunction satisfying $\left(-\mathrm{i}\hbar\frac{\partial}{\partial t} + H_1^{nQ}\right)\psi_{-n}^{nQ} = 0$ is given as

$$\psi_{-n}^{nQ} = \frac{1}{\rho\bar{u}_n^Q}. \tag{44}$$

For non-negative even integer n, it is clear that ψ_m^{nQ} $(m = -n, 0, 1, 2\ldots)$ are square-integrable.

Even for the simple harmonic oscillator of unit mass and frequency, since the shape of the probability density of a wavefunction could breathe and oscillate [14, 15], the region where $\Delta V/\hbar$ is different from 0 by a certain amount could breathe and oscillate. If we take the classical solutions as $\tilde{u}(t) = u(t) = \cos t$, $\tilde{v}(t) = v(t) = c\sin t\,(c \neq 0)$ and $x_p(t) = d\cos t$, the breathing and/or oscillating behaviour of the non-vanishing region appears when $c \neq 1$ and/or $d \neq 0$, respectively. For the choice $c = 1$ and $d = 0$, H_1^{nQ} becomes

$$H_1^{ns} = \frac{1}{2}(p^2 + x^2) - \hbar - \hbar^2\frac{\partial^2}{\partial x^2}\ln H_n\left(\mathrm{i}\frac{x}{\sqrt{\hbar}}\right). \tag{45}$$

Through the same choice of classical solutions, for non-negative even integer n, one can also find the eigenfunctions of H_1^{ns} as

$$\psi_m^{s,n} = -\sqrt{\frac{2}{\hbar}}\left[\sqrt{m+1}\mathrm{e}^{\mathrm{i}t}\psi_{m+1}^s + \sqrt{2}n\mathrm{i}\frac{H_{n-1}\left(\mathrm{i}\frac{x}{\sqrt{\hbar}}\right)}{H_n\left(\mathrm{i}\frac{x}{\sqrt{\hbar}}\right)}\psi_m^s\right] \qquad \psi_{-n}^{s,n} = \frac{1}{\bar{u}_n^s} \tag{46}$$

whose eigenvalues are $\left(m + \frac{1}{2}\right)\hbar$ and $-\left(n + \frac{1}{2}\right)\hbar$, respectively, as expected in [6].

3.2. A two-fold transformation

In this subsection, we will show that the two-fold application of the Darboux transformation is also possible for a general quadratic system. It will also be manifest that, in the two-fold transformation, the breathing and oscillating behaviour would still appear in the transformed systems of a simple harmonic oscillator.

By taking

$$u_1(t,x) = \psi_n(t,x) \qquad u_2(t,x) = \psi_{n+1}(t,x) \tag{47}$$

one can find that

$$\begin{aligned} W_2 &= \psi_n\psi'_{n+1} - \psi_{n+1}\psi'_n \\ &= \frac{1}{2^n n!\sqrt{2(n+1)}}\left(\frac{u-\mathrm{i}v}{\rho}\right)^{2n+1}\frac{1}{\rho}\sqrt{\frac{\Omega}{\hbar}}\psi_0^2(t,x)J_n(w) \end{aligned} \tag{48}$$

where

$$J_n(w) = H_n(w)\frac{\mathrm{d}}{\mathrm{d}w}H_{n+1}(w) - H_{n+1}(w)\frac{\mathrm{d}}{\mathrm{d}w}H_n(w) \tag{49}$$

with $w = \sqrt{\frac{\Omega}{\hbar}}\frac{x-x_p}{\rho}$. Making use of a recursive relation among the Hermite polynomials [16], one can show that $J_n(w) = 2H_n^2(w) + 2nJ_{n-1}(w)$. This relation, with the fact $J_0 = 2$, shows that $J_n(w)$ is positive definite for all x, which in fact has been implied in [13]. One can also find that

$$-\mathrm{i}\frac{\hbar}{2M}\frac{\partial^2}{\partial x^2}\ln\frac{W_2}{\bar{W}_2} = 4a + 2\frac{\dot{\rho}}{\rho}. \tag{50}$$

Equation (50) shows that α_2^Q can be given in accordance with equation (14) as

$$\alpha_2^Q(t) = \rho^2(t) \tag{51}$$

up to a multiplicative constant.

With the W_2, the two-fold transformation can, therefore, be applied to the quadratic system to give the transformed Hamiltonian

$$H_2^{nQ} = H_Q + 2\frac{\hbar\Omega}{M\rho^2} - \frac{\hbar^2}{M}\frac{\partial}{\partial x^2}\ln J_n(w). \tag{52}$$

For the case of $M(t) = 1$ and $a(t) = 0$, H_2^{nQ} reduces to the Hamiltonian found in [12]. Making use of equations (10), (13) and (16), it is also possible to find the wavefunctions of the system of H_2^{nQ}. For non-vanishing W_2, it turns out that the same set of classical solutions must be used for both u_1 and u_2 in equation (47). However, in obtaining wavefunctions, a different set of classical solutions could be used for ψ in equation (10), as in the Darboux transformation of the model.

4. A quadratic system with an inverse-square interaction

In this section, we will consider the application of the Darboux transformation for the quadratic system with an inverse-square interaction described by the Hamiltonian

$$H_{\text{in}} = \frac{p^2}{2M(t)} - a(t)(xp + px) + \frac{1}{2}M(t)c(t)x^2 + \frac{g}{M(t)}\frac{1}{x^2} \tag{53}$$

defined on the half line $x > 0$, where g is a constant. The system of H_{in} is related to the system described by the Hamitonian

$$H_{\text{in}}^s = \frac{p^2}{2} + \frac{1}{2}x^2 + \frac{g}{x^2}$$

through a unitary relation [15]. If $u(t), v(t)$ denote the homogeneous solutions of equation (21) as in the previous section, and $\rho(t)$ and Ω are defined by equation (24), a wavefunction of the system of H_{in} is given, for non-negative integer n, as

$$\begin{aligned}\psi_n^{\text{in}}(t,x) = {} & \left(\frac{4\Omega}{\hbar\rho^2}\right)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2}\left(\frac{u-\mathrm{i}v}{\rho}\right)^{(2n+\alpha+1)}\left(\frac{\Omega x^2}{\hbar\rho^2}\right)^{(2\alpha+1)/4} \\ & \times \exp\left[-\frac{x^2}{2\hbar}\left(\frac{\Omega}{\rho^2} - \mathrm{i}M\frac{\dot\rho}{\rho} - 2\mathrm{i}Ma\right)\right] L_n^\alpha\left(\frac{\Omega x^2}{\hbar\rho^2}\right)\end{aligned} \tag{54}$$

where α is defined through the relation $g = \frac{1}{2}\left(\alpha + \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)\hbar^2$, and L_n^α is the Laguerre polynomial [16]. For $\alpha > -1$, $\psi_n^{\text{in}}(t,x)$ is square-integrable on the half line.

Through similar procedures used in the previous section, one can find the auxiliary function

$$\begin{aligned}v_n^{\text{in}} = {} & \frac{1}{\sqrt{\rho}}\left(\frac{u+\mathrm{i}v}{\rho}\right)^{(2n+\alpha+1)}\left(\frac{\Omega_0 x^2}{\hbar\rho^2}\right)^{(2\alpha+1)/4} \\ & \times \exp\left[\frac{x^2}{2\hbar}\left(\frac{\Omega_0}{\rho^2} + \mathrm{i}M\frac{\dot\rho}{\rho} + 2\mathrm{i}Ma\right)\right] L_n^\alpha\left(-\frac{\Omega_0 x^2}{\hbar\rho^2}\right)\end{aligned} \tag{55}$$

satisfying

$$\left(-\mathrm{i}\hbar\frac{\partial}{\partial t} + H_{\text{in}}\right) v_n^{\text{in}} = 0. \tag{56}$$

From the properties of the Laguerre polynomial, v_n^{in} has no zero on the half line, and it is easy to see that

$$-\mathrm{i}\frac{\hbar}{2M}\frac{\partial^2}{\partial x^2}\ln\frac{v_n^{\text{in}}}{\bar v_n^{\text{in}}} = 2a + \frac{\dot\rho}{\rho}. \tag{57}$$

Equations (14) and (57) show that $\alpha_1^{\text{in}}(t)$ can be given as

$$\alpha_1^{\text{in}}(t) = \rho(t) \tag{58}$$

up to a normalization constant. The Darboux transformation can therefore be carried out, with $u_1 = v_n^{\text{in}}(t,x)$, to find the Hamiltonian of a solvable model as

$$H_1^{n,\text{in}} = H_{\text{in}} - \frac{\hbar^2}{2M}\frac{\partial^2(v_n^{\text{in}}\bar v_n^{\text{in}})}{\partial x^2}. \tag{59}$$

In implementing the formulae (10), (13) and (16), different sets of homogeneous solutions of equation (21) can be used for $v_n^{\text{in}}(= u_1)$ and $\psi_m^{\text{in}}(= \psi)$ to find the general expression of the wavefunctions of the $H_1^{n,\text{in}}$ system, as in the general quadratic system of the previous section. For simplicity, however, we only consider the case that the same set $\{u(t), v(t)\}$ of the homogeneous solutions is used in v_n^{in} and ψ_m^{in}. The wavefunctions $\psi_m^{n,\text{in}}$ of the $H_1^{n,\text{in}}$ system, if we adopt the notation $L_{-1}(y) = 0$, is given as

$$\psi_m^{n,\mathrm{in}} = -2\frac{\rho}{x}\sqrt{m(m+\alpha)}\left(\frac{u-\mathrm{i}v}{\rho}\right)^2 \psi_{m-1}^{\mathrm{in}} + 2\frac{\rho}{x}\left[(m-n-1)y + (n+\alpha)\frac{L_{n-1}^{\alpha}(-y)}{L_n^{\alpha}(-y)}\right]\psi_m^{\mathrm{in}} \tag{60}$$

for $m = 1, 2, 3, \ldots,$ where

$$y = \frac{\Omega x^2}{\hbar\rho^2}. \tag{61}$$

For $m = 0$, the wavefunction of the $H_1^{n,\mathrm{in}}$ system is given as

$$\psi_0^{n,\mathrm{in}} = 2\frac{\rho}{x}\left[-(n+1)y + (n+\alpha)\frac{L_{n-1}^{\alpha}(-y)}{L_n^{\alpha}(-y)}\right]\psi_0. \tag{62}$$

It is clear that $\psi_m^{n,\mathrm{in}}(t,x)$ is square-integrable on the half line for $\alpha > 0$. Though another formal solution of the Schrödinger equation of the $H_1^{n,\mathrm{in}}$ system can be found through equation (19), that solution turns out to be not square-integrable for $\alpha > 0$.

As implied in [13], for the two-fold application of the Darboux transformation, the auxiliary functions can be chosen as

$$u_1(t,x) = \psi_n^{\mathrm{in}}(t,x) \qquad u_2(t,x) = \psi_{n+1}^{\mathrm{in}}(t,x) \tag{63}$$

which gives

$$W_2^{\mathrm{in}} = 2\frac{n!}{(n+\alpha)!}\sqrt{\frac{n+1}{n+1+\alpha}}\left(\frac{u-\mathrm{i}v}{\rho}\right)^{4n+2}\left(\psi_0^{\mathrm{in}}\right)^2\frac{\Omega x}{\hbar\rho^2}K_n(y) \tag{64}$$

where

$$K_n(y) = L_n^{\alpha}(y)\frac{\mathrm{d}}{\mathrm{d}y}L_{n+1}^{\alpha}(y) - L_{n+1}^{\alpha}(y)\frac{\mathrm{d}}{\mathrm{d}y}L_n^{\alpha}(y). \tag{65}$$

Making use of a recurrence relation among the Laguerre polynomials, one can easily find that

$$K_n(y) = -\frac{1}{n+1}\left(L_n^{\alpha}(y)\right)^2 + \frac{n+\alpha}{n+1}K_{n-1}(y). \tag{66}$$

With the fact that $K_0 = -1$, (66) proves that $K_n(y)$ is negative-definite for all n, so that W_2^{in} does not vanish for all $0 < x < \infty$. The fact

$$-\mathrm{i}\frac{\hbar}{2M}\frac{\partial^2}{\partial x^2}\ln\frac{W_2^{\mathrm{in}}}{\bar{W}_2^{\mathrm{in}}} = 4a + 2\frac{\dot{\rho}}{\rho} \tag{67}$$

shows that the Hermiticity condition (14) of the Hamiltonian can be satisfied with

$$\alpha_2^{\mathrm{in}}(t) = \rho^2(t). \tag{68}$$

With W_2^{in}, from (17), the Hamiltonian of the transformed system can be found as

$$H_2^{\mathrm{in}} = H_{\mathrm{in}} + \frac{2\hbar^2(\alpha+1)}{Mx^2} + 2\frac{\hbar\Omega}{M\rho^2} - \frac{\hbar^2}{M}\frac{\partial^2}{\partial x^2}\ln K_n(y). \tag{69}$$

It may be possible, through the formulae (10), (13) and (16), to find the wavefunctions of the system of H_2^{in}.

5. Discussion

We have generalized the Darboux transformation to be applicable to a general one-dimensional time-dependent Hamiltonian system. The formalism for an *N*-fold application of the transformation has also been established. It has been shown that an Hermitian system can be found from the transformed system, if the auxiliary function(s) of the transformation could support the existence of a purely time-dependent function satisfying a certain condition (equation (14)). The formalisms have been applied to a general quadratic system and a quadratic system with an inverse-square interaction. As the potential difference between the original and the transformed systems is calculated from a (formal) solution of the Schrödinger equation, due to unitary relations responsible for the existence of coherent states in the system, the shape of the potential difference could oscillate according to a classical solution for the general quadratic system.

Since the potential difference is calculated from a solution of the Schrödinger equation, the difference depends on the Planck constant $\hbar$. In the cases considered, the magnitude of the potential difference is $O(\hbar)$. The range within which the potential difference effectively depends on the space coordinate would also be described in terms of $\hbar$, as has been shown explicitly in the examples considered. These features are not usual in the standard text book of quantum mechanics, and the potential difference appears as if it is a quantum correction which depends on the position.

It should also be mentioned that, for the systems considered in this paper, there may exist other choices of auxiliary functions which lead to Hermitian systems. Only for limited cases, however, may it be possible to find exact solutions of the Schrödinger equation of a time-dependent system. Even if the solutions are found, the auxiliary function should be chosen judiciously so that the transformed system could be Hermitian. As an example, $v_1^Q + v_2^Q$ is a solution of the Schrödinger equation of the general quadratic system, so that the Darboux transformation can be formally applied to it. However, with this solution, one cannot find $\alpha(t)$ satisfying (14).

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